Queue Length Distributions in a Markov Model of a Multistage Clocked Queueing Network

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Ultracomputer Note #161

June, 1989

ABSTRACT

In [4], we treated the problem of passage through a discrete-time clock-regulated multistage queueing network by modeling the input time series a_n to each queue as a Markov chain. We showed how to transform probability transition information from the input of one queue to the input of the next one in order to predict mean queue length at each stage. The Markov approximation is very good for $\rho = E(a_n) \leq \frac{1}{2}$, which is in fact the range of practical utility. Here we carry out a Markov time series input analysis to predict the stage to stage change in the probability distribution of queue length. The main reason for estimating the queue length distribution at each stage is to locate "hot spots", loci where unrestricted queue length would exceed queue capacitym and a quite simple expression is obtained for this purpose.

Supported in part by Department of Energy contract DE-FG02-88ER025053 and National Science Foundation grant DCR-8413359.

1. Introduction and Summary of Previous Results

The unit element of the network we consider [1] will be a 2×2 buffered switch, which we can regard as a system of two queues working in parallel, each one with a deterministic server of unit serving time. A message entering either of the two inputs of the switch goes with probability 1/2 to either of the two output queues of the switch. Each queue out-

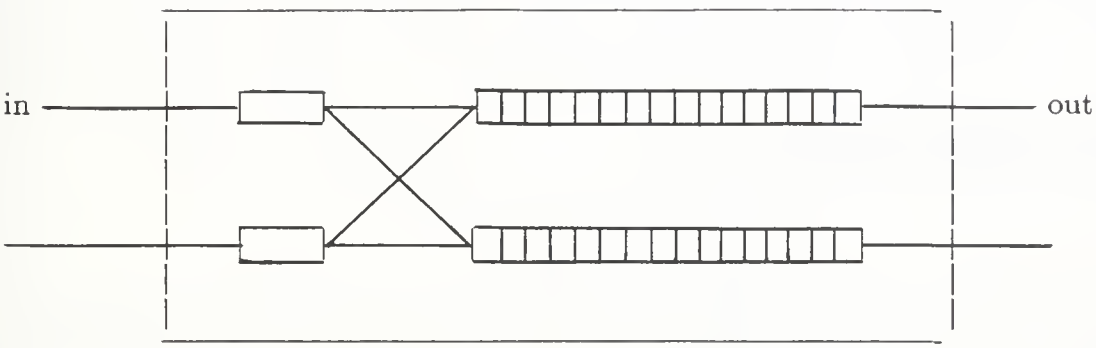

Fig. 1. 2×2 buffered switch.

puts one message per clock cycle unless it is empty. Any node in this network will then see a time series, each element of which signifies the presence or absence of an information packet to be transmitted to the next node. Thus, we are observing the transformation of a discrete time series by the basic processes of passage through the queue and of mixing.

Suppose the inputs to the switch at time n are denoted by I_n and I'_n, each taking the value 0 or 1. If we confine our attention to the upper channel, then according to the rule of mixing,

$$a_{n+1} = \gamma_n I_n + \gamma'_n I'_n \tag{1}$$

will arrive at the head of the upper queue, where

$$\gamma_n = \begin{cases} 0 & 1/2 \\ & \text{with probability} \\ 1 & 1/2 \end{cases} \tag{2}$$

and similarly for γ'_n. The streams $\{\gamma_n\}$ and $\{\gamma'_n\}$ are i.i.d. random variables. Now, if the

queue had length S_n to start, then at the end of the clock cycle

$$S_{n+1} = S_n + a_{n+1} - 1 + \delta_{n+1}$$

$$\text{where } \delta_{n+1} = \begin{cases} 1 & \text{if } S_n + a_{n+1} = 0 \\ 0 & \text{otherwise} \end{cases} , \tag{3}$$

or equivalently

$$S_{n+1} = \max\{S_n + a_{n+1} - 1, 0\} \tag{4}$$

Furthermore, the output, which is given by

$$O_{n+1} = 1 - \delta_{n+1} , \tag{5}$$

serves as input for the next stage. Evidently, the above can be taken over unchanged to the operation of the lower channel and associated queue. Although the outputs of the two channels are clearly correlated, matters may be arranged, e.g. in the Ω-network [1], so that they and their progeny never meet again as joint inputs, and so this correlation will not be investigated.

The picture we have in minde is that of a network of switches. This means that at a given switch

$$I_n = \overline{O}_n , \quad I'_n = \overline{O}'_n , \tag{6}$$

where $\overline{O}_n$ and $\overline{O}'_n$ are outputs from previous switches. We will be interested in steady

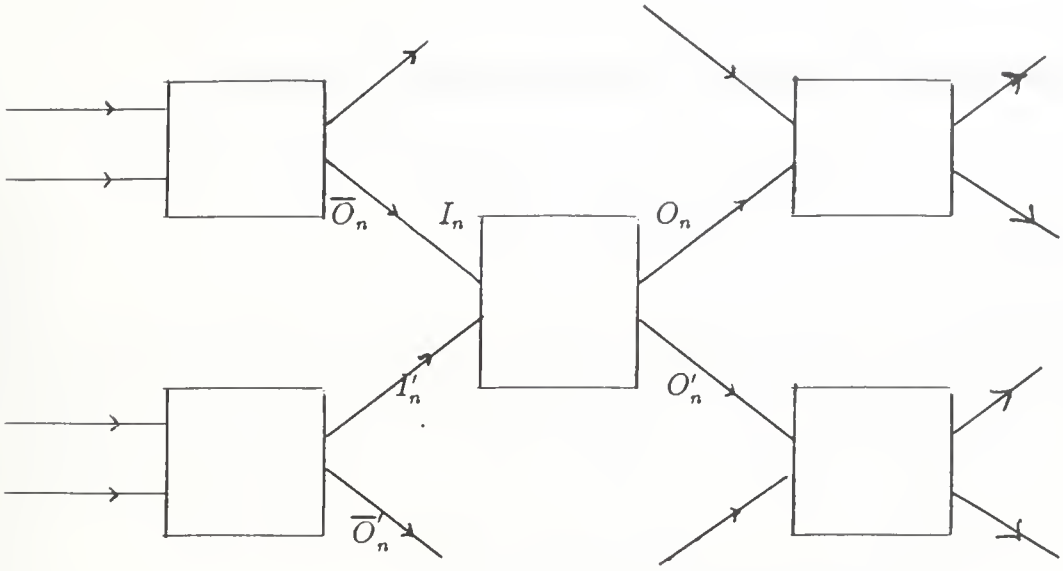

Fig. 2. Section of network.

state operation, described by the basic flow parameter

$$\rho = E\{I_n\} = E\{a_n\} \,, \tag{7}$$

in terms of which

$$f_k = P\{a_n = k\} = \binom{2}{k} (\frac{\rho}{2})^k (1 - \frac{\rho}{2})^{2-k} \tag{8}$$

More detailed information is supplied by the transition probabilities

$$\rho_0 = P\{I_n = 1 \; I_{n-1} = 0\} \,, \quad \rho_1 = P\{I_n = 1 \mid I_{n-1} = 0\} \tag{9}$$

Since $\rho = P\{I = 1\} = \rho_0(1 - \rho) + \rho_1\rho$, Eq. (9) is more conveniently written as

$$\rho_1 = 2(p + qr) \,, \quad \rho_0 = \frac{2p}{q - p}(q - p - 2qr) \tag{10}$$

where

$$p = \rho/2 \,, \quad q = 1 - \rho/2 \tag{11}$$

and r is a normalized departure from independence

$$r = \frac{\rho_1 - \rho}{2 - \rho} \,. \tag{12}$$

It is an easy exercise to show [4] that the input transition probabilities

$$Q_{kk'} = P\{a_n = k \mid a_{n-1} = k'\} \tag{13}$$

are then determined by (1) as

k \ k'	0	1	2
0	$(q+pr)^2$	$q(q+pr)(1-r)$	$q^2(1-r)^2$
1	$2p(q+pr)(1-r)$	$2pq(1-r)^2+r$	$2q(p+qr)(1-r)$
2	$p^2(1-r)^2$	$p(p+qr)(1-r)$	$(p+qr)^2$

Table 1. The transition matrix $Q_{kk'}$.

If we restrict ourselves to the case that $\{a_n\}$ is modeled to lie in the class of Markov time series, the above information allows us to use (3) to compute the steady state development of the random variable S_n at any stage without simultaneously developing the full time series $\{a_n\}$ — we simply follow $Q_{kk'}$, or equivalently, r, from stage to stage. In particular, we found [4] without difficulty that passage through a switch transforms r to

$$r^* = \frac{q-p}{4q}(p+2qr+pr^2)\,. \tag{14}$$

With the Markov model, we also found, with a certain amount of effort, that

$$E\{S_n\} = \frac{p^2}{q-p}\,\frac{1+r}{1-r}\,. \tag{15}$$

Comparsion with computer simulations showed this to be in general a lower bound to the exact result, but to be very accurate for $\rho < 1/2$, which is the practical domain of operation. Our objective now is to extend this result to the full distribution of S_n. The reason is a practical one: queues have finite capacity. If a message arrives at a queue which is full, a new mode of operation must be entered. It is thus important for design purposes to know how likely a given mode is to develop such a "hot spot", and this is what we will determine.

2. The Steady State Probability Distribution of Queue Length

Our objective now is to obtain

$$E_i(e^{tS}) \equiv \lim_{n\to\infty} E_i\{e^{tS_n}\} , \tag{16}$$

the generating function for the steady state queue size at the i^{th} stage of the system. Suppose that the input sequence $\{a_n\}$ to the i^{th} stage is given. Thus according to (3), and using $\delta_n^2 = \delta_n$, we have

$$\begin{aligned} e^{tS_n} &= e^{ta_n}\, e^{t(\delta_n - 1)}\, e^{tS_{n-1}} \\ &= e^{ta_n}(e^{-t} + (1-e^{-t})\delta_n)e^{tS_{n-1}} . \end{aligned} \tag{17}$$

But

$$\delta_n = \delta_{a_n,0}\,\delta_{S_{n-1},0} , \tag{18}$$

and so we can write

$$\begin{aligned} e^{tS_n} &= e^{t(a_n-1)}e^{tS_{n-1}} + (1-e^{-t})\delta_{a_n,0}\delta_{S_{n-1},0} \\ &= (e^{-t} + (1-e^{-t})T)e^{ta_n}\, e^{tS_{n-1}} , \end{aligned} \tag{19}$$

where T is the operator defined by

$$T\ h(t) = h(-\infty) \tag{20}$$

On iterating (19), it follows that

$$e^{tS_n} = \prod_{i=n}^{1} [(e^{-t} + (1-e^{-t})T)e^{ta_i}]e^{tS_0} . \tag{21}$$

Thus far, there is no approximation. Now however, we assume that the input time series is Markovian, with transition matrix given by Table 1 and parameter r appropriate to the stage in question. Hence

$$P\{a_n, \ldots, a_1\} = \sum_{a_0} \prod_{i=n}^{1} Q_{a_i a_{i-1}} p(a_0) , \tag{22}$$

and if the process is started at $S_0 = 0$, then (21) implies that

$$E\{e^{tS_n}\} = \sum_{a_n,\ldots,a_1,a_0} \prod_{i=n}^{1} [(e^{-t} + (1-e^{-t})T)e^{ta_i} Q_{a_i,a_{i-1}}] \cdot p(a_0) \quad . \tag{23}$$

Introducing the diagonal matrix

$$A_{aa'} = a\delta_{aa'} \ , \tag{24}$$

Eq. (23) takes on the concise form

$$E\{e^{tS_n}\} = \sum_{a,a'} [(e^{-t} + (1-e^{-t})T)e^{tA}\, Q]^n_{aa'} \cdot p(a') \ , \tag{25}$$

so that we must examine the properties of the operator

$$\Lambda = (e^{-t} + (1-e^{-t})T)e^{tA}\, Q \ . \tag{26}$$

Λ may be regarded as operating on the product space of input 3-vectors times power series in e^t. Λ is positive on this basis, and is stochastic:

$$\sum_{aa'} \Lambda_{aa'} P_{a'}(e^t)|_{t=0} = \sum_{aa'} e^{ta} Q_{aa'} P_{a'}(e^t)|_{t=0} = \sum_{a'} P_{a'}(1) \ . \tag{27}$$

It follows that the steady state limit $n \to \infty$ of (25) exists, and is determined by the eigenspaces of the maximal eigenvalue $\lambda = 1$ of Λ.

We now have

$$\lim_{n\to\infty} \sum_{aa'} (\Lambda^n)_{aa'} p(a') = K \sum_a f_a(t) \tag{28}$$

where $\{f_a(t)\}$ is the eigenvector belonging to $\lambda = 1$, and K a suitable constant. Since $E\{e^{tS_n}\}|_{t=0} = 1$, this means that

$$E\{e^{tS}\} = \sum_a f_a(t) \ / \ \sum_a f_a(0) \tag{29}$$

where

$$(e^{-t} + (1-e^{-t})T)e^{tA}\, Qf(t) = f(t) \ . \tag{30}$$

In order to solve (30), we rewrite it as

$$(I - e^{t(A-I)}Q)\, f(t) = (1 - e^{-t})\, T e^{tA}\, Q f(t) \tag{31}$$

and then define

$$g = T\, e^{tA}\, Q\, f(t) \tag{32}$$

so that

$$f(t) = (I - e^{t(A-I)}Q)^{-1}(1 - e^{-t})g\ . \tag{33}$$

Hence

$$\begin{aligned} g &= Te^{tA}Q(I - e^{t(A-I)}Q)^{-1}(1 - e^{-t})g \\ &= T(e^{t} - 1)\, e^{t(A-I)}Q(I - e^{t(A-I)}Q)^{-1}g \\ &= -T\, e^{t(A-I)}Q(I - e^{t(A-I)}Q)^{-1}g \\ &= -T((I - e^{t(A-I)}Q)^{-1} - I)g \\ &= g - T(I - e^{t(A-I)}Q)^{-1}g \end{aligned} \tag{34}$$

or simply

$$T(I - e^{t(A-I)}Q)^{-1}g = 0\ . \tag{35}$$

It is readily verified that

(36)

$$T(I-e^{t(A-I)}Q)^{-1} = \begin{pmatrix} 0 & Q_{01} & Q_{01}Q_{12} + Q_{01}(1 - Q_{11}) \\ 0 & -Q_{00} & Q_{10}Q_{02} - Q_{00}Q_{12} \\ 0 & 0 & -Q_{00}(1 - Q_{11}) - Q_{01}Q_{10} \end{pmatrix} \Big/ (-Q_{00}(1-Q_{11})-Q_{01}Q_{10})\ .$$

and so (35) implies that

$$g = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\ . \tag{37}$$

From (33), then,

$$\sum_a f_a(t) = \sum_a \Omega_{a0}^{adj}(1 - e^{-t})/\text{Det } \Omega \qquad (38)$$

$$\text{where } \Omega = I - e^{t(A-I)}Q$$

or, using the fact that Q is a stochastic matrix to eliminate the Q_{0a},

$$\sum_a f_a(t) = \begin{vmatrix} 1 & 1 & 1 \\ -Q_{10} & 1-Q_{11} & -Q_{12} \\ -e^t Q_{20} & -e^t Q_{21} & 1-e^t Q_{22} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 0 & -e^{-t} \\ -Q_{10} & 1-Q_{11} & -Q_{12} \\ -e^t Q_{20} & -e^t Q_{21} & 1-e^t Q_{22} \end{vmatrix} . \qquad (39)$$

On inserting the explicit $Q_{aa'}$ from Table I, we then have

$$\sum_a f_a(t) = C - D\, e^t \;/\; A - Be^t$$

$$\text{where } A = (1-Q_{20})(1-Q_{11}) - Q_{12}Q_{21} = (1-r)(q+pr)^2$$

$$B = Q_{21}(1-Q_{11}) + Q_{21}Q_{12} = (1-r)(p+qr)^2 \qquad (40)$$

$$C = 1 + Q_{10} - Q_{11} = (1-r)(1+2pr)$$

$$D = (Q_{22}-Q_{20})(1-Q_{11}) + (Q_{21}-Q_{20})Q_{12} + (Q_{22}-Q_{21})Q_{10}$$

$$= (1-r)r(2p+r)\ ,$$

and therefore conclude from (29) that

$$E\{e^{tS}\} = (q-p)[(1+2pr) - r(2p+r)e^t] \;/\; [(q+pr)^2 - (p+qr)^2 e^t]\ . \qquad (41)$$

3. Discussion

An immediate result of (41) is the expected length of the queue:

$$E\{S\} = \frac{\partial}{\partial t} \ell n E\{e^{tS}\}|_{t=0} = \frac{p^2}{q-p} \frac{1+r}{1-r} , \tag{42}$$

as well as the variance

$$\sigma^2\{S\} = \frac{\partial^2}{\partial t^2} \ell n E\{e^{tS}\}|_{t=0} = \frac{p^2 q^2}{(q-p)^2} \left(\frac{1+r}{1-r}\right)^2 - 2p^2 \frac{r}{(1-r)^2} . \tag{43}$$

For successive stages in a network, we have seen, (14) that starting at $r_1 = 0$ corresponding to independent previous output [2,3]

$$r_{i+1} = \frac{q-p}{4q}(p + 2qr_i + pr_i^2) . \tag{44}$$

The mean queue length can thus be followed through a network in Markov approximation, and compared with computer simulation. The results, as we have mentioned, are quite good for $\rho < 0.5$, and this is true for the variance as well. This encourages us to use (41) to compute the full queue length distribution. We have immediately from (41)

$$P\{S = k\} = \text{coef } e^{tk} \text{ in } E\{e^{tS}\} = \begin{cases} (q-p)\frac{1+2pr}{(q+pr)^2} , & k = 0 \\ (q-p)\left[\frac{1+2pr}{(q+pr)^2} - \frac{r(r+2p)}{(p+qr)^2}\right]\left(\frac{p+qr}{q+pr}\right)^{2k} , & k > 0, \end{cases} \tag{45}$$

where at the i^{th} stage, r is set equal to r_i of (44).

It is easily seen that for $\rho < \frac{1}{2}$, r_i of (44) increases monontonically with i to a value which, with negligible error, is given by

$$r_\infty \sim 2r_2 = \frac{p}{2q} \frac{1-2p}{1+2p} . \tag{46}$$

The corresponding distribution (45), for small p, reduces to

$$P\{S = k > 0\} \sim \frac{4}{9}\left(\frac{p(3-p)}{2}\right)^{2k} . \tag{47}$$

We therefore have a simple estimate for the frequency P at which a queue of capacity k near the termination of the network saturates under mean flow $\rho = 2p$.

References

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